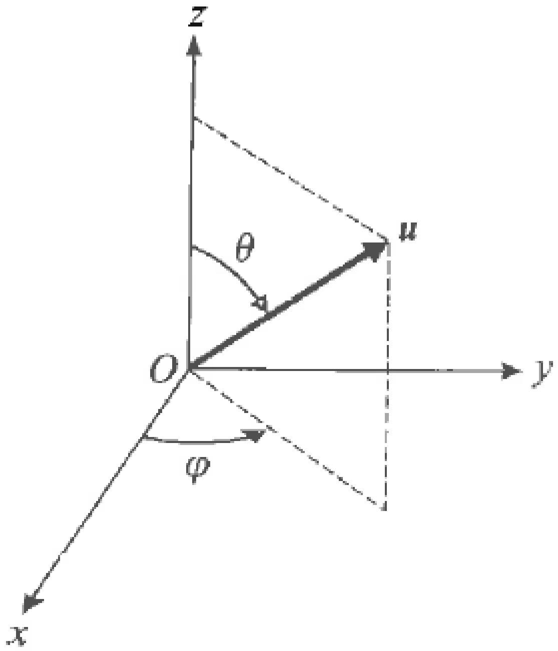
Stern-Gerlach Device Oriented at an Arbitrary Angle

Orient the SG machine such that the magnetic field gradient is in the $\hat{\boldsymbol{u}}$ direction



$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{x}}\,\sin\boldsymbol{\theta}\,\cos\boldsymbol{\varphi} + \hat{\boldsymbol{y}}\,\sin\boldsymbol{\theta}\,\sin\boldsymbol{\varphi} + \hat{\boldsymbol{z}}\,\cos\theta$$

A beam of neutral Ag atoms sent through a device oriented in this way will produce a beam in this spin state:

$$\hat{\boldsymbol{S}}_{\boldsymbol{u}} = \vec{\hat{\boldsymbol{S}}}\cdot\hat{\boldsymbol{u}} = \hat{\boldsymbol{S}}_{\boldsymbol{x}}\sin\boldsymbol{\theta}\,\cos\boldsymbol{\varphi} + \hat{\boldsymbol{S}}_{\boldsymbol{y}}\sin\boldsymbol{\theta}\,\sin\boldsymbol{\varphi} + \hat{\boldsymbol{S}}_{\boldsymbol{z}}\cos\theta$$

$$\hat{\boldsymbol{S}}_{\boldsymbol{u}} = \frac{\hbar}{\mathbf{2}}\begin{pmatrix}\cos\theta & \sin\boldsymbol{\theta}\,\boldsymbol{e}^{-\boldsymbol{i\varphi}}\\ \sin\boldsymbol{\theta}\,\boldsymbol{e}^{+\boldsymbol{i\varphi}} & -\cos\theta\end{pmatrix}$$

The eigenvalues of the $\hat{\boldsymbol{S}}_{\boldsymbol{u}}$ operator are $\pm\hbar/\mathbf{2}$. The "up" and "down" eigenfunctions are:

$$|\uparrow\rangle_{\boldsymbol{u}} = \cos\frac{\theta}{2}\,e^{-i\varphi/2}|\uparrow\rangle_{\boldsymbol{z}} + \sin\frac{\theta}{2}\,e^{+i\varphi/2}|\downarrow\rangle_{\boldsymbol{z}}$$

$$|\downarrow\rangle_{\boldsymbol{u}} = -\sin\frac{\theta}{2}\,e^{-i\varphi/2}|\uparrow\rangle_{\boldsymbol{z}} + \cos\frac{\theta}{2}\,e^{+i\varphi/2}|\downarrow\rangle_{\boldsymbol{z}}$$

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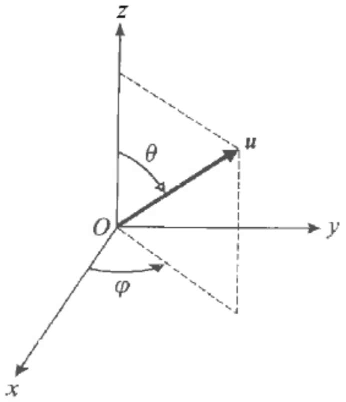
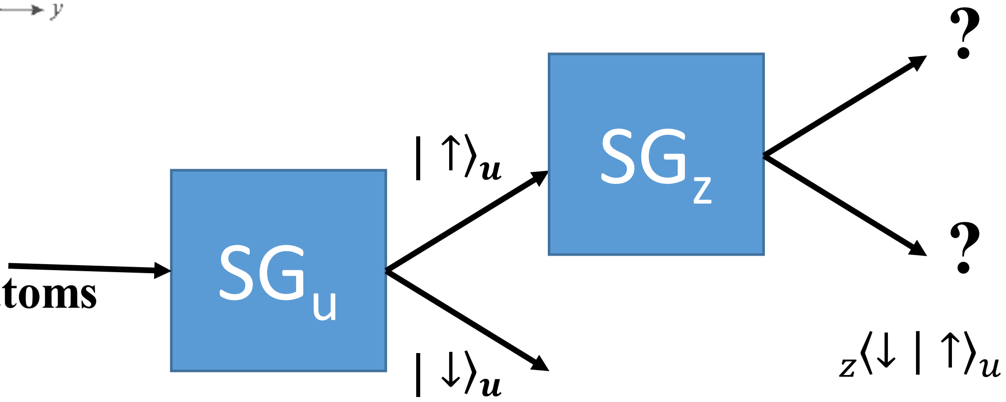
Use the first SG machine to create a beam of Ag atoms in the $|\uparrow\rangle_{\boldsymbol{u}}$ eigenstate

Now use a second SG machine to perform an $\hat{\boldsymbol{S}}_{\boldsymbol{z}}$ measurement of this beam. What is the outcome?



$${}_z\langle\uparrow|\uparrow\rangle_u = \cos\frac{\theta}{2}\, e^{-i\varphi/2} \qquad \left|{}_z\langle\uparrow|\uparrow\rangle_u\right|^2 = \boldsymbol{cos^2}\frac{\theta}{2}$$


Un-polarized beam of Ag atoms

SG$_u$

$|\uparrow\rangle_{\boldsymbol{u}}$

$|\downarrow\rangle_{\boldsymbol{u}}$

SG$_z$

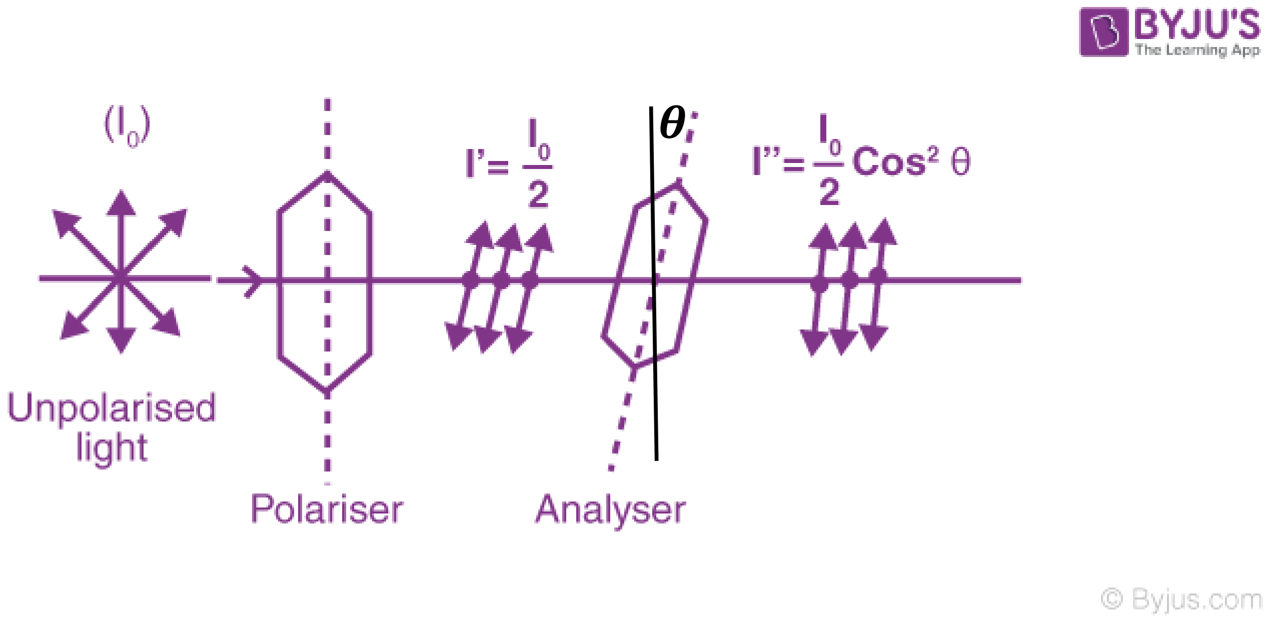
?

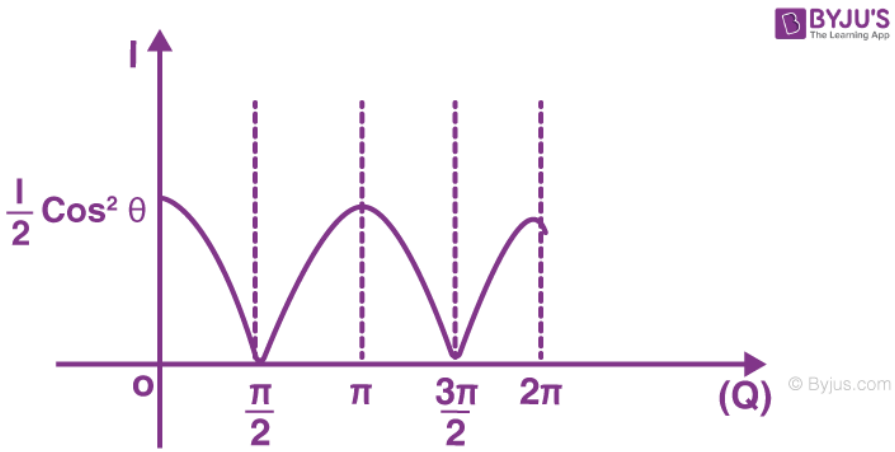
?

$${}_z\langle\downarrow|\uparrow\rangle_u = \sin\frac{\theta}{2}\, e^{+i\varphi/2} \qquad \left|{}_z\langle\downarrow|\uparrow\rangle_u\right|^2 = \boldsymbol{sin^2}\frac{\theta}{2}$$

Example: Suppose $\boldsymbol{\theta = \pi/2}$ and $\boldsymbol{\varphi = 0}$ ($\hat{\boldsymbol{u}} = \hat{\boldsymbol{x}}$), then $\left|{}_z\langle\uparrow|\uparrow\rangle_u\right|^2 = \frac{1}{2}$ and $\left|{}_z\langle\downarrow|\uparrow\rangle_u\right|^2 = \frac{1}{2}$

Note that this is different from Malus's Law for optical polarization (polarizer and analyzer)





https://byjus.com/jee/malus-law/#:~:text=Malus'%20law%20states%20that%20the,transmission%20axes%20of%20the%20analyzer.